Design of asymmetric operators using a weighted least-squares approximation

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Summary

A spatial convolution operator is designed so that it is stable and accurately matches the exact operator in the wavenumber domain in a desired wavenumber range. This can be done by a weighted least-squares optimization. Several applications require the use of an asymmetric operator. By introducing an asymmetric weight function, the properties of the operator can be influenced to reach an optimal design.

Introduction

In recent years, several methods have been developed to transform operators in the (k_1, f) domain to a spatial convolution operator in the (x_1, f) domain. For one-way wavefield extrapolation, Holberg (1988) and Thorbecke (1997) constructed spatial operators which are stable in a recursive extrapolation scheme. Migration in the (x_1, f) domain has the advantage that laterally varying media can be considered. A short operator should be designed to restrict computation time.

In most cases the convolution operators are symmetric. However, there are several applications for which asymmetric operators are preferred. For example, for applications where only a very small number of data points is available (in the same order as the number of operator points), artifacts are introduced for the greater part of the output points, since the operator acts outside the array of data points. This can be resolved by designing asymmetric operators which act on the total array of data points for each output point.

In this paper we discuss the design of asymmetric operators. The theory is set up in a general sense, i.e., for any application in which the use of asymmetric operators is required. One could think of wavefield extrapolation in anisotropic media with a tilted symmetry axis (Zhang et al., 2001), elastic wavefield decomposition, etc. The theory then is applied for the design of asymmetric wavefield extrapolation operators for imaging obstacles in the soil ahead of a tunnel boring machine.

Theory

Transformation of an operator in the (k_1, f) domain to the (x_1, f) domain has to be done in an optimal way so that it results in a stable spatial convolution operator. Over a desired wavenumber interval, its wavenumber spectrum has to be equal or at least close to the exact operator. This interval depends on the relative lateral position of the output point for which the convolution result will be calculated. In discrete form, the forward spatial transform can be written as

$$\tilde{W}(n\Delta k_1) = \Delta x_1 \sum_{m=M_1}^{M_2} \exp(jn\Delta k_1 m\Delta x_1) W(m\Delta x_1) \qquad \text{for} \quad N_1 \leq n \leq N_2. \tag{1}$$

The length of the spatial convolution operator is presented by $M_1 ... M_2$ and $N_1 ... N_2$ denotes the length of the Fourier transform. In matrix notation this becomes

$$\tilde{\mathbf{W}} = \mathbf{\Gamma W} \tag{2}$$

or

$$\begin{bmatrix} \tilde{W}(N_1\Delta k_1) \\ \vdots \\ \tilde{W}(0) \\ \vdots \\ \tilde{W}(N_2\Delta k_1) \end{bmatrix} = \mathbf{\Gamma} \begin{bmatrix} W(M_1\Delta x_1) \\ \vdots \\ W(0) \\ \vdots \\ W(M_2\Delta x_1) \end{bmatrix}, \tag{3}$$

with

$$\mathbf{\Gamma} = \Delta x_1 \begin{bmatrix} e^{(jN_1\Delta k_1 M_1 \Delta x_1)} & \cdots 1 \cdots & e^{(jN_1\Delta k_1 M_2 \Delta x_1)} \\ \vdots & \ddots \vdots \ddots & \vdots \\ 1 & \cdots 1 \cdots & 1 \\ \vdots & \ddots \vdots \ddots & \vdots \\ e^{(jN_2\Delta k_1 M_1 \Delta x_1)} & \cdots 1 \cdots & e^{(jN_2\Delta k_1 M_2 \Delta x_1)} \end{bmatrix}. \tag{4}$$

$\tilde{\mathbf{W}}$ is an approximation of the exact phase-shift operator, while $\mathbf{W}$ is the desired short spatial operator. To make sure that the spatial operator is zero outside its working length, the number of samples in the (k_1, f) domain must be greater or equal to the number of receivers. This means that Eq. 3 has more equations than unknowns and should be solved using an approximation procedure like the weighted least-squares method. The solution of the weighted least-squares operator is given by (Thorbecke, 1997)

$$\mathbf{W} = \left[\mathbf{\Gamma}^H \tilde{\mathbf{\Lambda}} \mathbf{\Gamma}\right]^{-1} \mathbf{\Gamma}^H \tilde{\mathbf{\Lambda}} \tilde{\mathbf{W}} \tag{5}$$

in which the weight function $\tilde{\mathbf{\Lambda}}$ is a diagonal matrix defined as

$$\tilde{\Lambda}_{mn} = w(n\Delta k_1)\delta_{mn}. \tag{6}$$

The weight function $w(n\Delta k_1)$ should be chosen in such a way that the desired part of the wavenumber spectrum gets a weight of one. Outside this interval, the spectrum is suppressed by a low weight factor to result in a stable behavior of the operator. The size of the interval is depending on the angles where the operator should be close to the exact operator. By changing the parameters of the weight function, the operator can be designed in an optimal sense.

Asymmetric wavefield extrapolation operators

The theory in the previous section applies to any symmetric or asymmetric operator (wavefield decomposition, (an)isotropic wave field extrapolation, etc.). From here onward, we consider wavefield extrapolation in isotropic media. In Thorbecke (1997), the weighted least-squares approximation is described for symmetric spatial operators. A short spatial operator for extrapolation in an isotropic medium is designed to reduce computation time. Therefore a symmetric operator can easily be determined around each depth point. In this case, it holds that

$$M_1 = -M_2, \tag{7}$$

$$W(m\Delta x_1) = W(-m\Delta x_1), \tag{8}$$

$$\tilde{W}(n\Delta k_1) = \tilde{W}(-n\Delta k_1). \tag{9}$$

When the spatial operator is designed asymmetric, the corresponding extrapolation operator in the (k_1, f) domain will become also asymmetric.

The phase-shift operator is a one-way extrapolation operator in the (k_1, f) domain that is used for recursive extrapolation schemes. This operator will be used as reference for the optimization of the operators in the next section. The phase-shift operator for a 2-dimensional medium is given by (Gazdag, 1978), (Berkhout, 1985), (Wapenaar and Berkhout, 1989)

$$\tilde{W}(k_1, \Delta x_3, f) = \exp(-jk_3\Delta x_3), \tag{10}$$

with

$$k_3 = \begin{cases} \sqrt{k^2 - k_1^2} & k_1^2 \le k^2 \\ -j\sqrt{k_1^2 - k^2} & k_1^2 > k^2 \end{cases} \tag{11}$$

in which k is defined as ω/c with c being the propagation velocity. For wavenumbers larger than k, the wavefield becomes evanescent.

Results

One of the applications for which an asymmetric operator could be used is tunnel boring in soft soils. The tunnel boring machine is constructed to excavate through sands and clays and hitting hard obstacles should be avoided. Therefore a source and receivers are installed on the cutter wheel of the tunnel boring machine to image the subsurface in front of the machine (Swinnen et al., 1999). In this case, the number of receivers that can be used and their aperture is restricted by the diameter of the tunnel boring machine. Transformation of the spatial data to the (k_1, f) domain will introduce a lot of artifacts because of the small number of traces. Therefore migration should be carried out by a spatial convolution operator. The length of the operator is determined by the number of receivers. In an extrapolation step each point for which the extrapolated wavefield has to be calculated, determines the asymmetry of the operator. The operator should be accurate over a range of angles which is selected via the weight function $w(n\Delta k_1)$ (Eq. 6).

The setup for this discussion can be seen in Fig. 1 where operators with a 9-point length are designed. For output depth point $x_1^{(0)}$, the operator will be symmetric. Since the subsurface is considered to be locally homogeneous, the operator for output depth point $x_1^{(i)}$ will be the mirror image of the operator for output depth point $x_1^{(-i)}$. The following parameters are used: $\Delta x_1 = 0.5$ m, $\Delta x_3 = 0.15$ m, $f = 50$ Hz and $c = 150$ m/s (typical shear-wave velocity for soft soils).

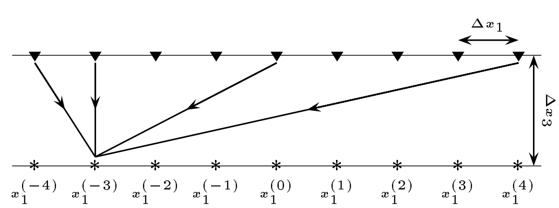

Fig. 1: Setup used for the design of the asymmetric spatial operators

The spatial convolution operator for output position $x_1^{(-3)}$ is shown in Fig. 2. The shape of the operator is not so smooth. Important in the weighted least-squares approximation is the k_1 - spectrum, plotted in Fig. 3 together with the exact operator. The estimated extrapolation operator is really close to the exact situation in the area of interest, the positive angles and very small negative angles. The largest error is made at the transition to the not important negative angles. This overshoot is suppressed as good as possible but this will always be at the expense of accuracy in the important part of the spectrum. Outside this range, the operator is stable.

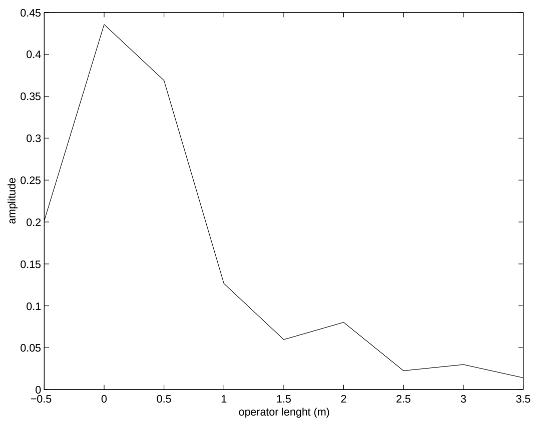

Fig. 2: Spatial convolution operator for output position $x_1^{(-3)}$

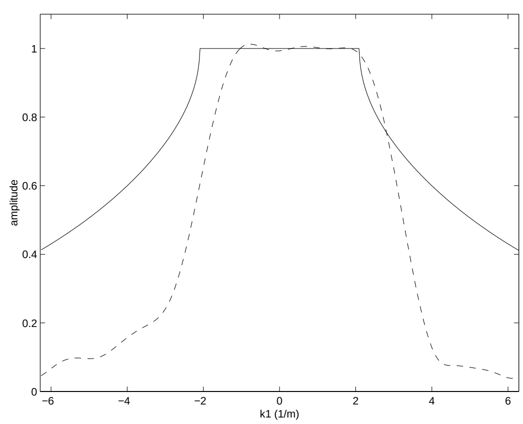

Fig. 3: Amplitude of the k_1 - spectrum of the operator for output position $x_1^{(-3)}$ (solid line is exact operator)

In Fig. 4, a set of operators is shown within a frequency range from 0 to 200 Hz. All operators are accurate for positive and small negative angles and stable outside this range. For frequencies higher than 100 Hz, the larger negative angles are less suppressed but they still show a stable behavior.

The relative lateral position of the point for which the operator has to be designed has a large influence on the

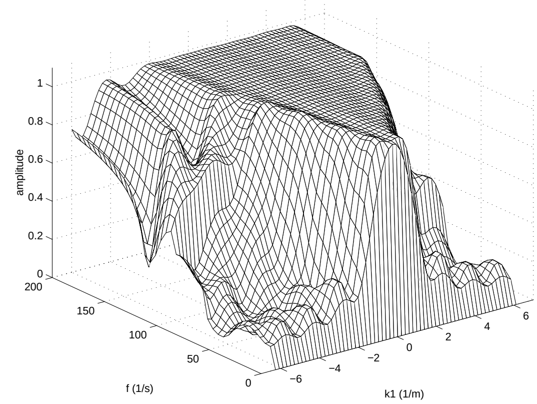

Fig. 4: Amplitudes of the (k_1, f) - spectra of the operators for output position $x_1^{(-3)}$ with frequency ranging from 0 to 200 Hz

accuracy of the operator. Fig. 5 shows the k_1 - spectra of the extrapolation operators for output positions $x_1^{(0)}$ (symmetric operator), $x_1^{(-2)}$ and $x_1^{(-3)}$, together with the exact phase shift operator. The symmetric operator fits the exact operator accurately for all angles. The other two curves are accurate over the positive and small negative angles. The range for output point $x_1^{(-2)}$ is of course a bit larger than for output point $x_1^{(-3)}$. At negative angles, the asymmetric operator for output point $x_1^{(-3)}$ shows a small overshoot that is harder to suppress without decreasing the accuracy in the important range of angles.

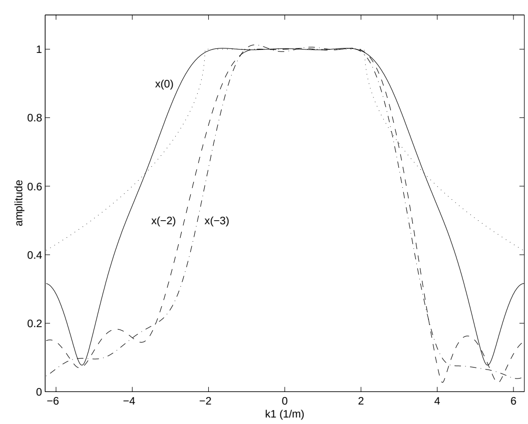

Fig. 5: k_1 - spectrum of the operators for output positions $x_1^{(0)}$, $x_1^{(-2)}$ and $x_1^{(-3)}$; the influence of the asymmetry of the operator (dotted line is exact operator)

For positive angles, the amplitude of the difference between the designed operators and the exact operator ($|\tilde{W}_{design} - \tilde{W}_{exact}|$) is plotted in Fig. 6 for the three curves in Fig. 5. It can be seen that the error of the complex operators in the (k_1, f) domain is very small.

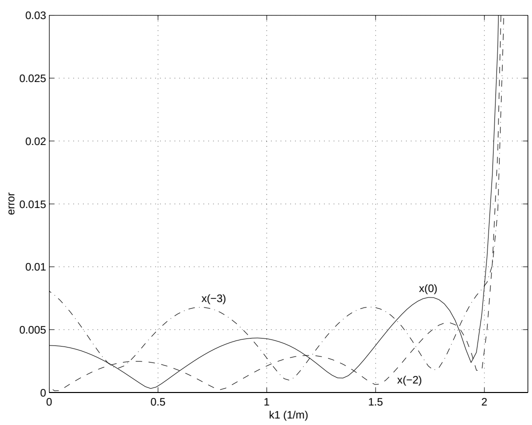

Fig. 6: Error for the operators for output positions $x_1^{(0)}$, $x_1^{(-2)}$ and$x_1^{(-3)}$ for positive angles

The small effect of the overshoot that could be seen in Fig. 5 is extreme for the outermost points $x_1^{(4)}$ and $x_1^{(-4)}$. The k_1 - spectrum for point $x_1^{(-4)}$ can be seen in Fig. 7. The accuracy in the area of positive angles is not as good as in the case of the operators in Fig. 5 (the amplitude axis has different scale) and the overshoot is unacceptable. The overshoot can be suppressed to an acceptable level, but the error in the important area of the k_1 - spectrum will increase largely. An operator for these outermost points should be designed using other methods like non-linear optimization techniques (Zhang et al., 2001).

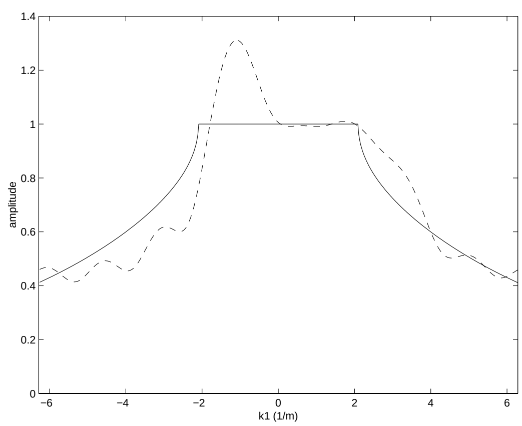

Fig. 7: k_1 - spectrum of the operator for output position $x_1^{(-4)}$

Conclusions

The design of asymmetric spatial convolution operators using a weighted least-squares procedure results in a good approximation of the exact operator in the desired k_1 range. The accuracy highly depends on the relative lateral position of the output point for which the operator has to be optimized. When the operator shifts further from the symmetric case, the accuracy decreases. For spatial operators where only positive or only negative angles are considered, the overshoot becomes too large to suppress with the current method.

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