From Reflection to Transmission

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Abstract

Using the reciprocity theorem a relation between the seismic reflection response and transmission data can be derived. This relation is used to calculate transmission data from reflection data measured at the surface. In this presentation the steps involved in this calculation are explained in detail and possible pitfalls are discussed. At the end a simple example is given to illustrate the discussed procedure.

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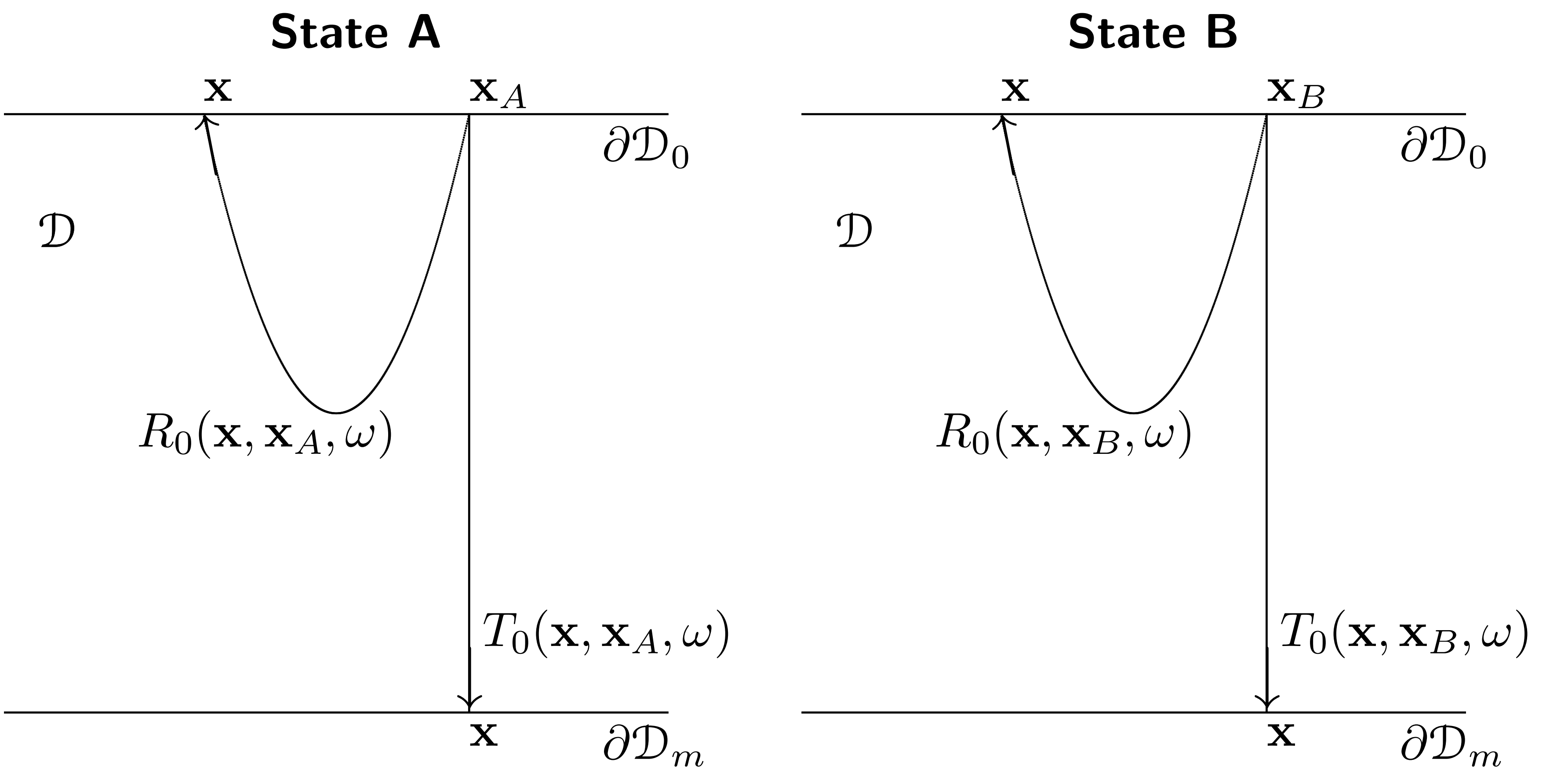
- The Theorem
- Structure of $\mathbf{T}$
- Matrices $\mathbf{R}^H\mathbf{R}$
- Diagonal Matrix $\mathbf{\Lambda}$
- Simple examples
- Practical problems
- Future work

One-way reciprocity relations of the correlation type



One-way reciprocity relations of the correlation type

$$\int_{\partial\mathcal{D}_0} \{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\} d^2\mathbf{x}_H = \int_{\partial\mathcal{D}_m} \{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\} d^2\mathbf{x}_H$$

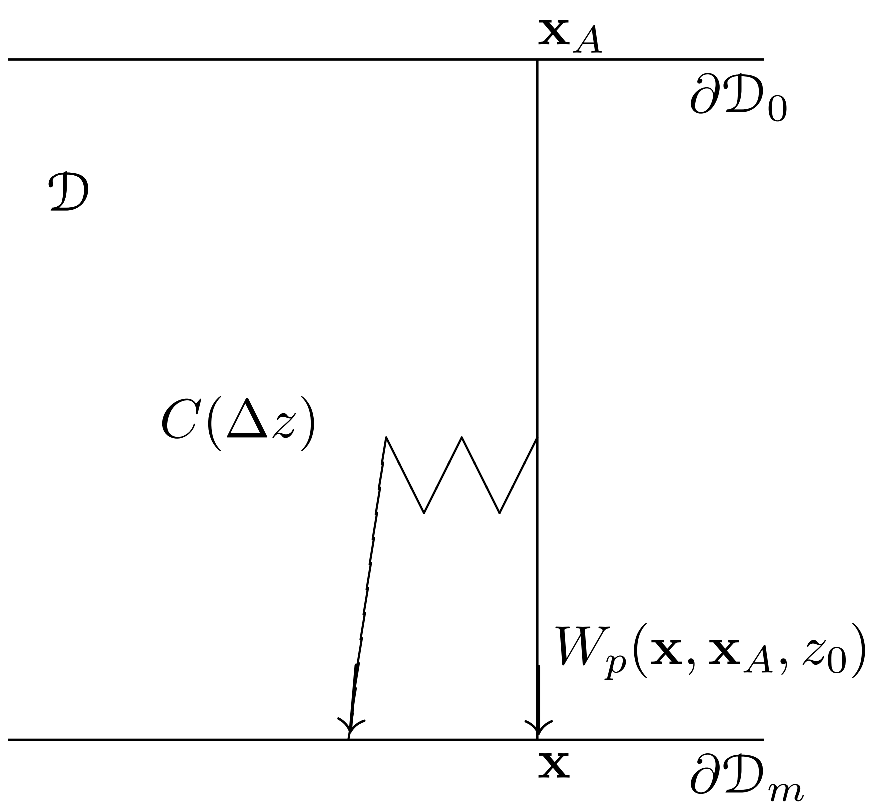
the medium parameters in both states A and B are identical, lossless and 3-D inhomogeneous and the domain $\mathcal{D}$ is source free.

One-way reciprocity relations of the correlation type

Surface $\partial\mathcal{D}_0$		
Field	**State A**	**State B**
P^+	$\delta(\mathbf{x}_H - \mathbf{x}_{H,A})s_A(\omega)$	$\delta(\mathbf{x}_H - \mathbf{x}_{H,B})s_B(\omega)$
P^-	$R_0(\mathbf{x}, \mathbf{x}_A, \omega)s_A(\omega)$	$R_0(\mathbf{x}, \mathbf{x}_B, \omega)s_B(\omega)$
Surface $\partial\mathcal{D}_m$		
P^+	$T_0(\mathbf{x}, \mathbf{x}_A, \omega)s_A(\omega)$	$T_0(\mathbf{x}, \mathbf{x}_B, \omega)s_B(\omega)$
P^-	0	0

$$\delta(\mathbf{x}_{H,A} - \mathbf{x}_{H,B}) - \int_{\partial\mathcal{D}_0} R_0^*(\mathbf{x}, \mathbf{x}_A) R_0(\mathbf{x}, \mathbf{x}_B) d^2\mathbf{x}_H = \int_{\partial\mathcal{D}_m} T_0^*(\mathbf{x}, \mathbf{x}_A) T_0(\mathbf{x}, \mathbf{x}_B) d^2\mathbf{x}_H$$

Structure of T



Structure of T

Transmission response is written as a flux normalized generalized primary propagator in the $\omega - p$ domain:

$$T_0(p, \mathbf{x}_A, z_m, z_0) = W_g(p, \mathbf{x}_A, z_m, z_0)$$
$$T_0(p, \mathbf{x}_A, z_m, z_0) = W_p(p, \mathbf{x}_A, z_m, z_0)C(p, \mathbf{x}_A, \Delta z)$$

with W_p the primary propagator and C the coda operator and $\Delta z = z_m - z_0$.

$$W_p(p, \mathbf{x}_A, z_m, z_0) = \exp(-j\omega \int_{z_0}^{z_m} (c^{-2}(z) - p^2)^{\frac{1}{2}} dz)$$
$$W_p^*(p, \mathbf{x}_A, z_m, z_0)W_p(p, \mathbf{x}_A, z_m, z_0) \approx 1.0$$
$$C(p, \mathbf{x}_A, \Delta z) = \exp(-\mathcal{A}(p)\Delta z)$$

Matrices $\mathbf{R}^H\mathbf{R}$

Replace Integral with Matrix notation.

$$\int_{\partial\mathcal{D}_0} R_0^*(\mathbf{x},\mathbf{x}_A) R_0(\mathbf{x},\mathbf{x}_B) d^2\mathbf{x}_H \equiv \boldsymbol{R}_0^H(\mathbf{x},\mathbf{x}_A)\boldsymbol{R}_0(\mathbf{x},\mathbf{x}_B)$$

$$= \big(R_0^*(\mathbf{x}_0,\mathbf{x}_A)\ldots R_0^*(\mathbf{x}_i,\mathbf{x}_A)\ldots R_0^*(\mathbf{x}_N,\mathbf{x}_A)\big)\begin{pmatrix} R_0(\mathbf{x}_0,\mathbf{x}_A) \\ \vdots \\ R_0(\mathbf{x}_i,\mathbf{x}_A) \\ \vdots \\ R_0(\mathbf{x}_N,\mathbf{x}_A) \end{pmatrix}$$

Matrices $\mathbf{R}^H\mathbf{R}$

Including multiple shot positions:

$$\mathbf{R} = \begin{pmatrix} R(\mathbf{x}_{r,0},\mathbf{x}_{s,0}) & R(\mathbf{x}_{r,0},\mathbf{x}_{s,1}) & \dots & R(\mathbf{x}_{r,0},\mathbf{x}_{s,N}) \\ R(\mathbf{x}_{r,1},\mathbf{x}_{s,0}) & R(\mathbf{x}_{r,0},\mathbf{x}_{s,1}) & \dots & R(\mathbf{x}_{r,0},\mathbf{x}_{s,N}) \\ \vdots & \vdots & \ddots & \vdots \\ R(\mathbf{x}_{r,M},\mathbf{x}_{s,0}) & R(\mathbf{x}_{r,0},\mathbf{x}_{s,1}) & \dots & R(\mathbf{x}_{r,M},\mathbf{x}_{s,N}) \end{pmatrix}$$

represents a fixed spread acquisition geometry.

Matrices $\mathbf{R}^H\mathbf{R}$

Using the matrices the integral equations becomes:

$$\mathbf{T}^H\mathbf{T} = \mathbf{I} - \mathbf{R}^H\mathbf{R}$$

$$\mathbf{T}^H\mathbf{T} = (\mathbf{W_p C})^H\mathbf{W_p C} = \mathbf{C}^H\mathbf{C}$$

$$\mathbf{C}^H\mathbf{C} = \mathbf{I} - \mathbf{R}^H\mathbf{R}$$

For fixed spread geometry both matrices are symmetric due to the reciprocity relation between source and receiver positions.

Eigenvalues $\boldsymbol{\Lambda}$

We need to resolve **C** from $\mathbf{C}^H\mathbf{C}$ and make the assumption that:

$$\mathbf{C} = \mathbf{L}\boldsymbol{\Lambda}_c\mathbf{L}^H$$

$$\boldsymbol{\Lambda}_c(p) = \exp\{-\underset{\sim}{\mathbf{A}}(p)\Delta z_{tot}\}$$

$$\boldsymbol{\Lambda}_c(p) = \begin{pmatrix} e^{-\mathcal{A}(p_1)\Delta z_{tot}} & 0 & \dots & 0 \\ 0 & e^{-\mathcal{A}(p_2)\Delta z_{tot}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{-\mathcal{A}(p_N)\Delta z_{tot}} \end{pmatrix}$$

Note, the coda was defined as $C(p, \Delta z) = \exp(-\mathcal{A}(p)\Delta z)$

Eigenvalues $\mathbf{\Lambda}$

$$\mathbf{C}^H\mathbf{C} = \mathbf{L}\mathbf{\Lambda}_c^H\mathbf{\Lambda}_c\mathbf{L}^H$$

$$\mathbf{\Lambda}_c^H\mathbf{\Lambda}_c = \exp\{-2\Re\{\underset{\sim}{\mathbf{A}}\}\Delta z_{tot}\}$$

$$\mathbf{\Lambda}_c^H\mathbf{\Lambda}_c = \begin{pmatrix} e^{-2\Re\{\mathcal{A}_1\}\Delta z_{tot}} & 0 & \dots & 0 \\ 0 & e^{-2\Re\{\mathcal{A}_2\}\Delta z_{tot}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{-2\Re\{\mathcal{A}_N\}\Delta z_{tot}} \end{pmatrix}$$

Eigenvalues $\boldsymbol{\Lambda}$

Coupling between $\mathbf{C}^H\mathbf{C}$ and $\mathbf{R}^H\mathbf{R}$

$$\mathbf{C}^H\mathbf{C} = \mathbf{I} - \mathbf{R}^H\mathbf{R}$$
$$\mathbf{C}^H\mathbf{C} = \mathbf{I} - \mathbf{L}\boldsymbol{\Lambda}_r\mathbf{L}^H$$
$$\mathbf{L}\boldsymbol{\Lambda}_c^H\boldsymbol{\Lambda}_c\mathbf{L}^H = \mathbf{L}[\mathbf{I} - \boldsymbol{\Lambda}_r]\mathbf{L}^H$$
$$\exp\{-2\mathcal{R}\{\underset{\sim}{\mathbf{A}}\}\Delta z_{tot}\} = \mathbf{I} - \boldsymbol{\Lambda}_r$$
$$\mathcal{R}[\underset{\sim}{\mathbf{A}}] = -\frac{1}{2\Delta z_{tot}}\ln\{\mathbf{I} - \boldsymbol{\Lambda}_r\}$$

Note, the use of $\mathbf{I}$.

Eigenvalues $\mathbf{\Lambda}$

The coda operator $\underset{\sim}{\mathbf{A}}(p,\omega)$ can be retrieved from its real part if the assumption is made that the elements $A_l(\omega)$ of $\underset{\sim}{\mathbf{A}}(p,\omega)$ are the Fourier transforms of causal filters in the time domain. Then the following relation (Hilbert Transform) holds:

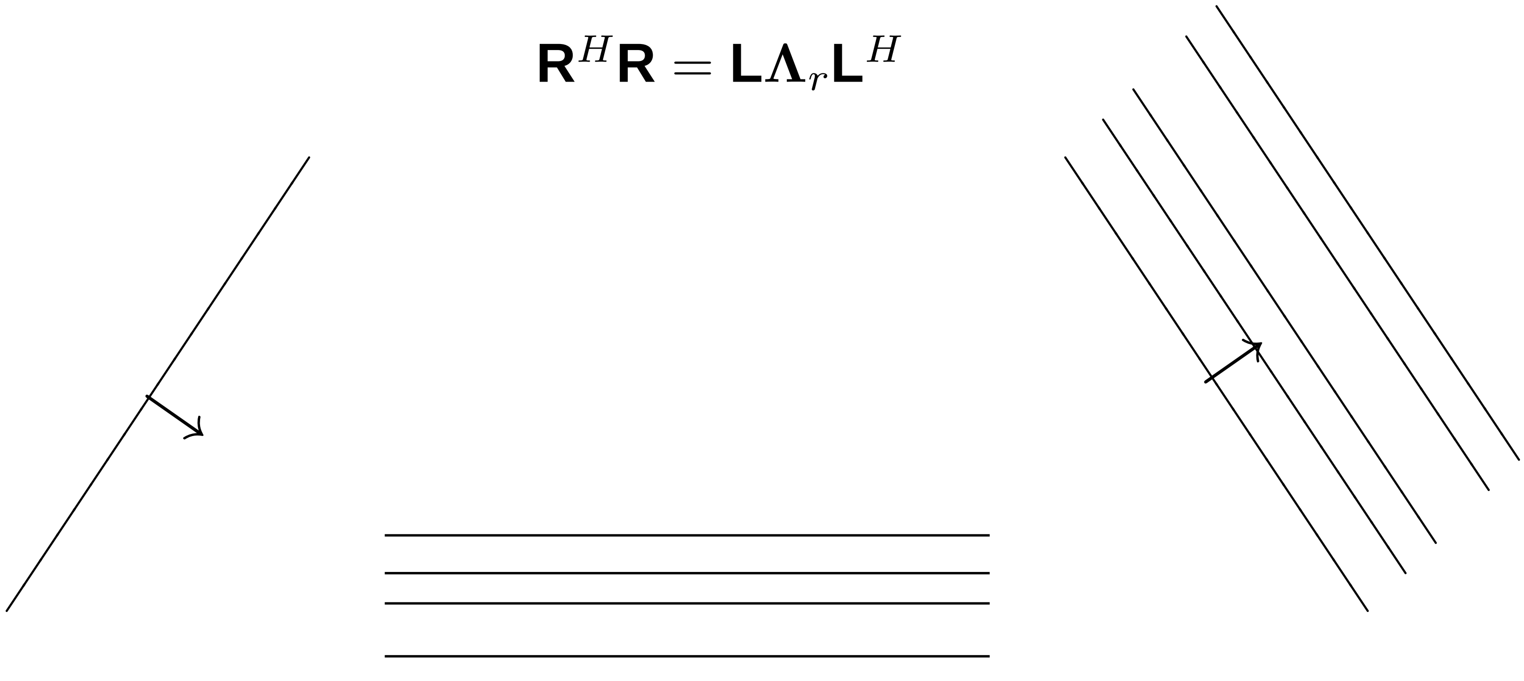
$$A(\omega) = \Re\{A(\omega)\} + \frac{1}{j\pi}\int_{-\infty}^{+\infty} \frac{\Re\{A(\omega')\}}{\omega - \omega'} d\omega'$$

In practice $\mathcal{A}(p,\omega)$ is transformed back to time with a complex to complex Fourier transform $a_r(p,t) = \mathcal{F}_{\omega\rightarrow t}\{\mathcal{A}(p,\omega)\}$ and multiplied with the Heaviside function to get the causal signal.

The coda can now be calculated with: $C(p,\omega,\Delta z) = \exp(-\mathcal{A}(p,\omega)\Delta z)$

Eigenvalues Λ

$$\mathbf{R}^H\mathbf{R} = \mathbf{L}\mathbf{\Lambda}_r\mathbf{L}^H$$



Simple examples

Starting with the measured reflection data compute:

$$\mathbf{R}^H\mathbf{R} = \begin{pmatrix} R^*(\mathbf{x}_{r,0}, \mathbf{x}_{s,0}) & \dots & R^*(\mathbf{x}_{r,M}, \mathbf{x}_{s,0}) \\ \vdots & \ddots & \vdots \\ R^*(\mathbf{x}_{r,0}, \mathbf{x}_{s,N}) & \dots & R^*(\mathbf{x}_{r,M}, \mathbf{x}_{s,N}) \end{pmatrix} \begin{pmatrix} R(\mathbf{x}_{r,0}, \mathbf{x}_{s,0}) & \dots & R(\mathbf{x}_{r,0}, \mathbf{x}_{s,N}) \\ \vdots & \ddots & \vdots \\ R(\mathbf{x}_{r,M}, \mathbf{x}_{s,0}) & \dots & R(\mathbf{x}_{r,M}, \mathbf{x}_{s,N}) \end{pmatrix}$$

For a 1 dimensional medium this reduces to:

$$\mathbf{R}^H\mathbf{R} = \begin{pmatrix} R^*R(0) & \dots & R^*R(N) \\ \vdots & \ddots & \vdots \\ R^*R(N) & \dots & R^*R(0) \end{pmatrix}$$

where R^*R is the auto-correlation function of the reflection response R.

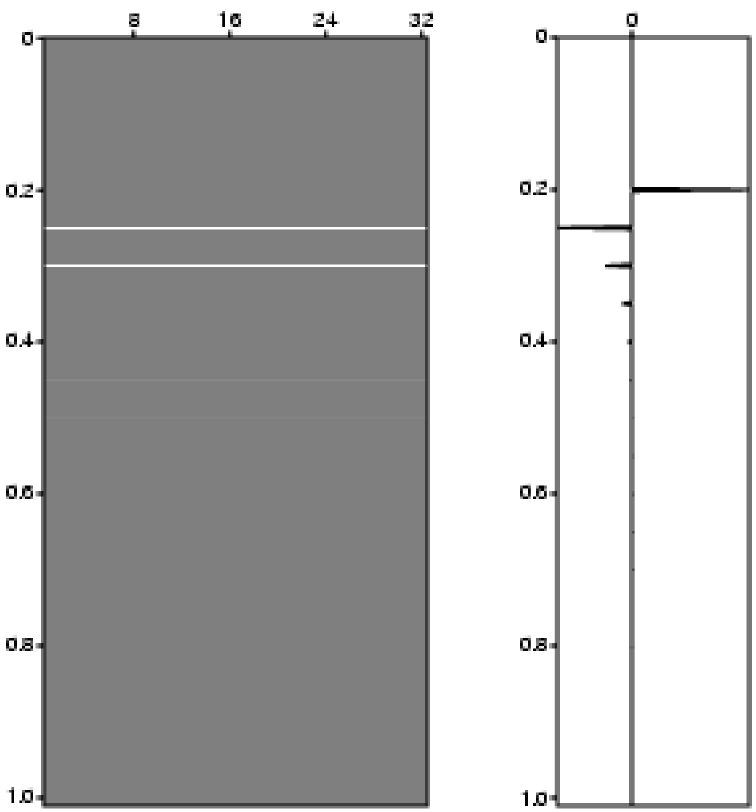
Simple examples: plane wave

$\mathbf{R}^H\mathbf{R}$ is a Circulant matrix for 1D sources (plane waves) in 1D media. The eigenvalues of a circulant matrix are obtained by taking the Fourier transform of the first column, so $A_l(\omega) = \tilde{A}_l(k_{x,l}, \omega)$.

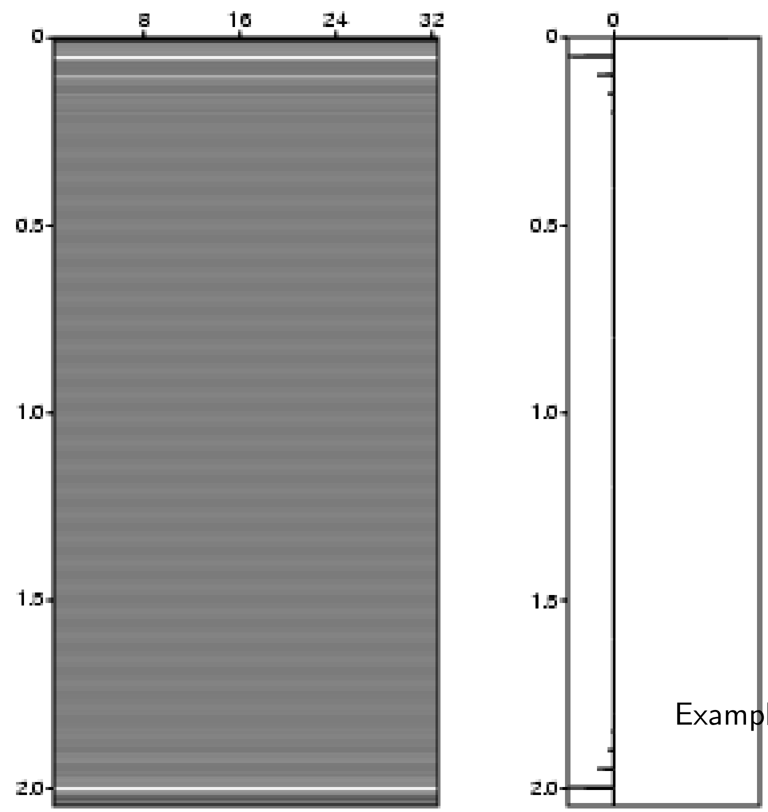
3 layer medium 1000-4000-1000 m/s thickness 100 m: $4000/200 = 0.05$ s. internal multiple train.

Simple examples: plane wave

shot-record **R**



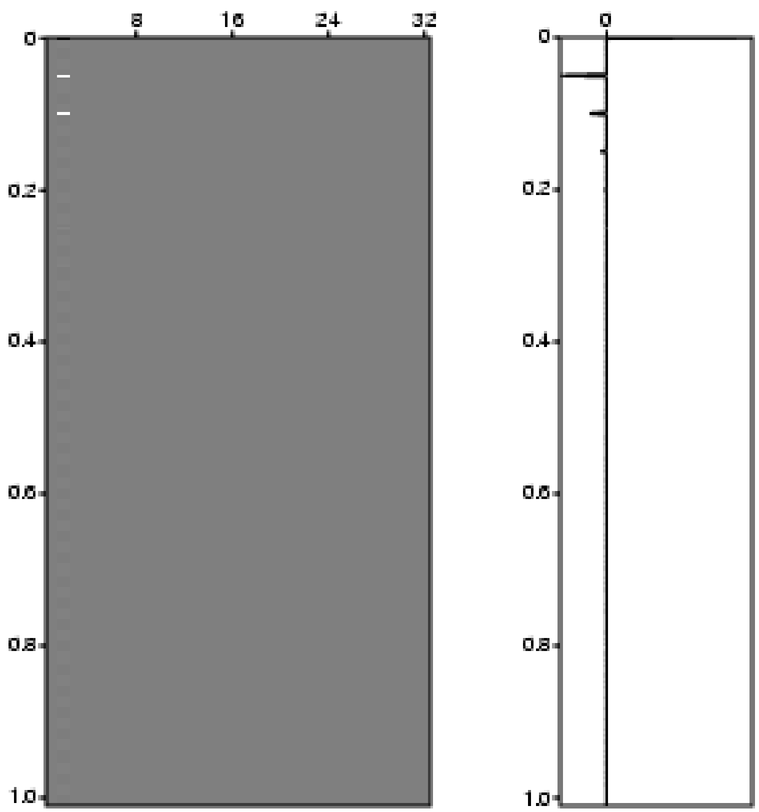
auto-correlation $\mathbf{R}^H\mathbf{R}$



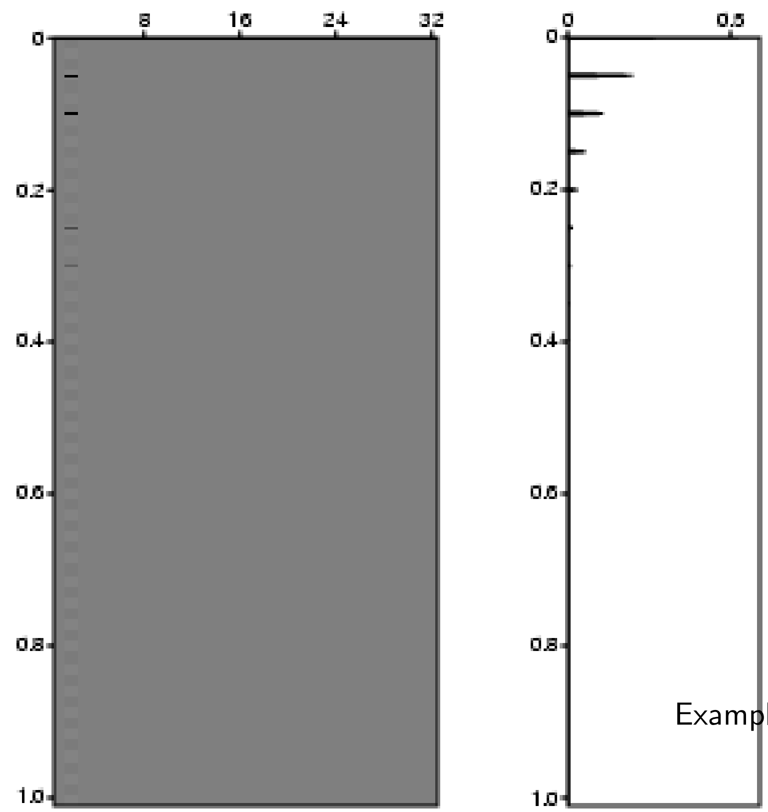
Simple examples: plane wave

Eigenvalues calculated with Lapack routine

diagonal $\mathbf{\Lambda}_r$ 2'nd trace



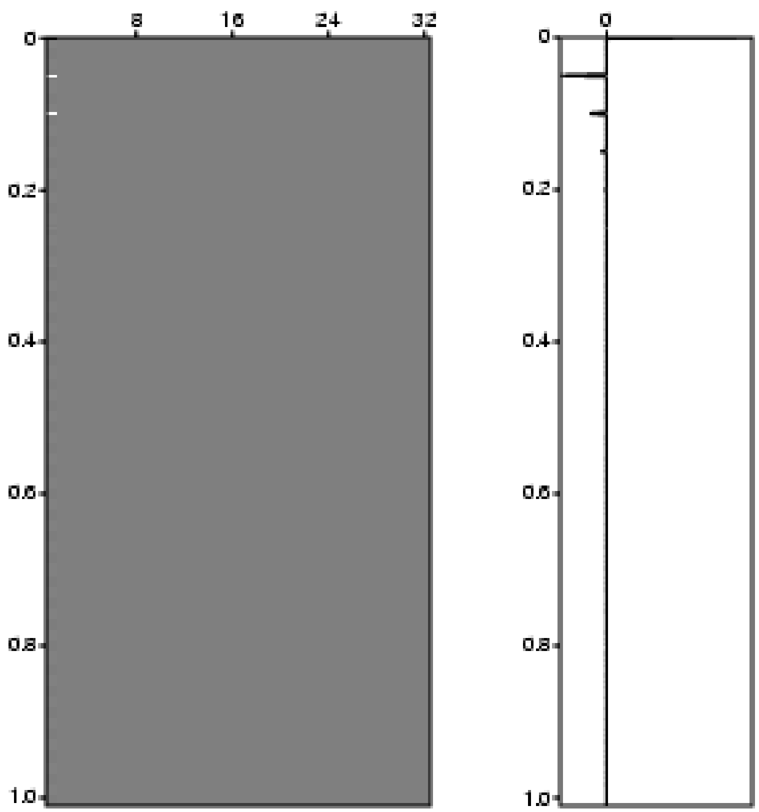
coda **C**



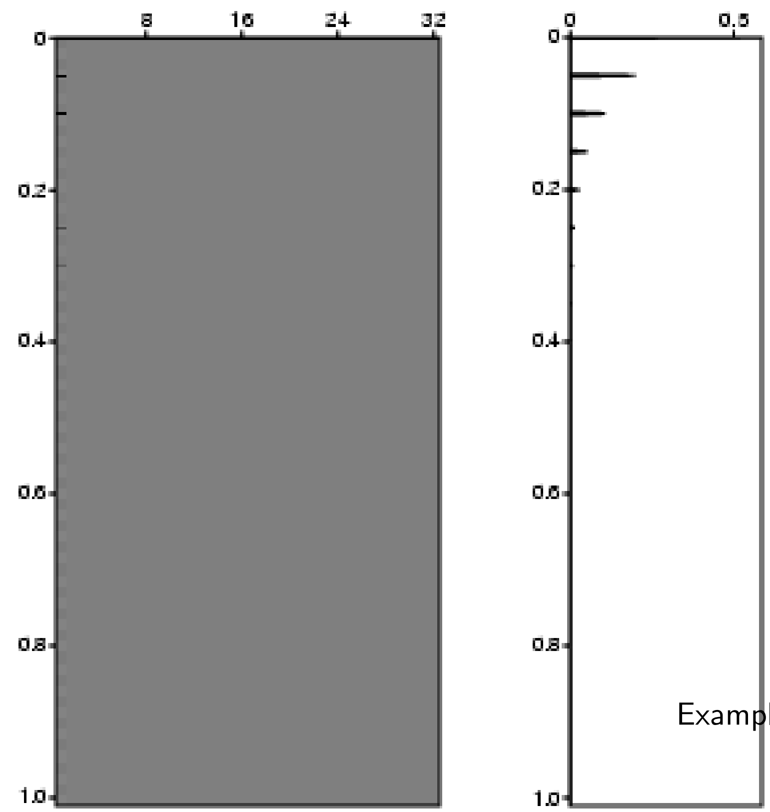
Simple examples: plane wave

Eigenvalues calculated with FFT routine

diagonal $\mathbf{\Lambda}_r$ 1'st trace



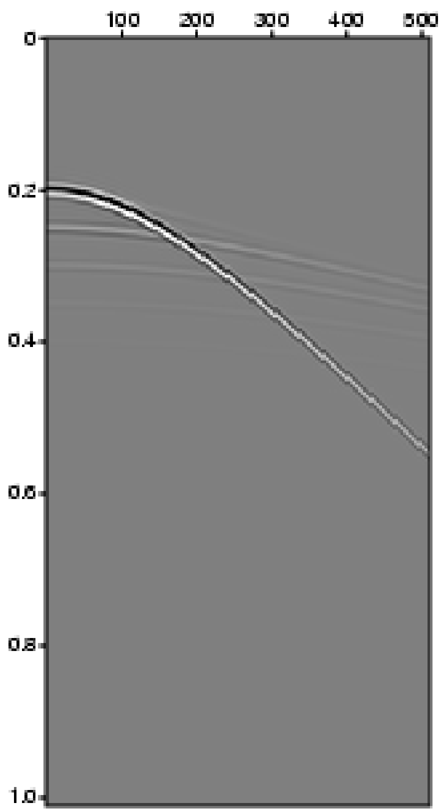
coda **C**



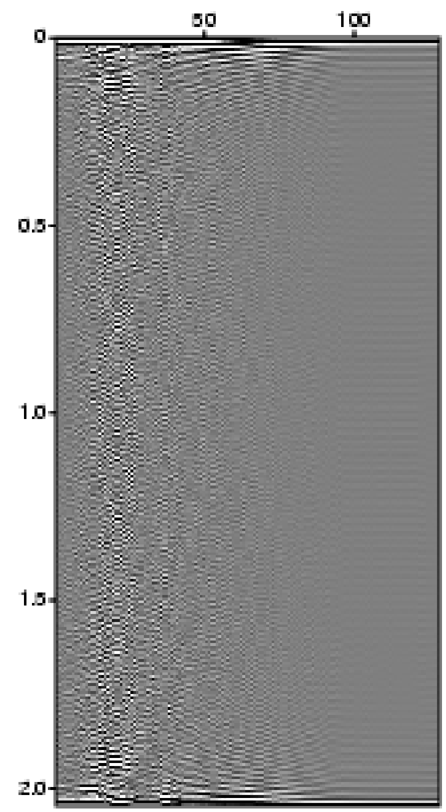
Simple examples: point source

Eigenvalues calculated with Lapack routine

shot record **R**

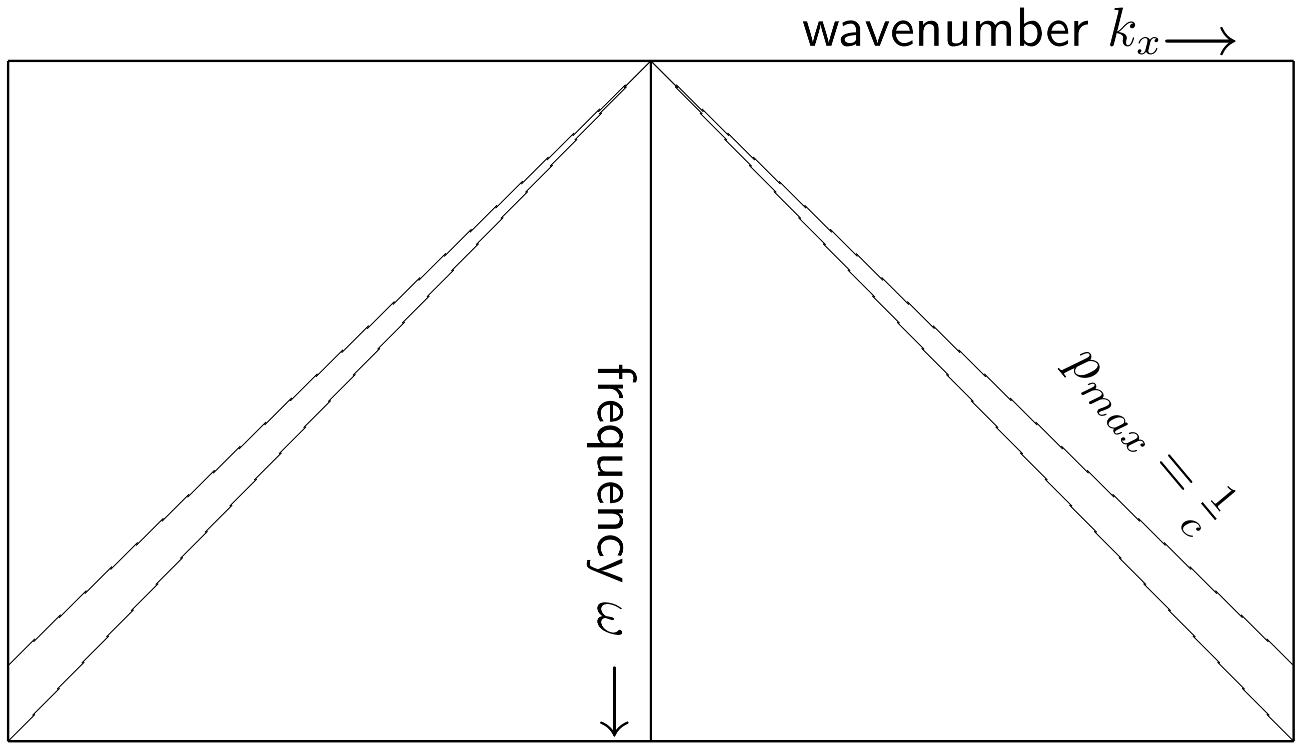


diagonal



Simple examples: plane wave

Scaling from wavenumber k_x to ray-parameter p with $\frac{1}{\omega}(k_x = p\omega)$.



Practical problems

$$\mathbf{R}(\mathbf{x},\omega) \Rightarrow \mathbf{R}^H\mathbf{R} \Rightarrow \mathbf{L}\boldsymbol{\Lambda}_r\mathbf{L}^H \Rightarrow \boldsymbol{\Lambda}_r(p) \Rightarrow \mathbf{A}(p,\omega) \Rightarrow \exp\{-\mathbf{A}(p,\omega)\} \Rightarrow \mathbf{T}(p,\omega)$$

- Computation of $\mathbf{I} - \mathbf{R}^H\mathbf{R}$
- Ordering of eigenvalues (per ω) and mapping to k_x
- Check eigenvectors: $\mathbf{L}^H\mathbf{L} = \mathbf{I}$
- Eigenvector analysis for 2D media
- Correct scaling of amplitudes

- Windowing of R; below $\partial\mathcal{D}_m$ heterogeneous media

- Construction of W_g from C and W_p

Future work

- Solve the practical problems
- 2D media examples
- close loop $R \to R_0 \to T_0 \to T \to R$